3.4 Particle motion

Dynamics definitions[a]

Newtonian force	$\boldsymbol{F} = m\ddot{\boldsymbol{r}} = \dot{\boldsymbol{p}}$	(3.63)	$\boldsymbol{F}$ force m mass of particle $\boldsymbol{r}$ particle position vector
Momentum	$\boldsymbol{p} = m\dot{\boldsymbol{r}}$	(3.64)	$\boldsymbol{p}$ momentum
Kinetic energy	$T = \frac{1}{2}mv^2$	(3.65)	T kinetic energy v particle velocity
Angular momentum	$\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{p}$	(3.66)	$\boldsymbol{J}$ angular momentum
Couple (or torque)	$\boldsymbol{G} = \boldsymbol{r} \times \boldsymbol{F}$	(3.67)	$\boldsymbol{G}$ couple
Centre of mass (ensemble of N particles)	$\boldsymbol{R}_0 = \frac{\sum_{i=1}^{N} m_i \boldsymbol{r}_i}{\sum_{i=1}^{N} m_i}$	(3.68)	$\boldsymbol{R}_0$ position vector of centre of mass m_i mass of ith particle $\boldsymbol{r}_i$ position vector of ith particle

[a] In the Newtonian limit, $v \ll c$, assuming m is constant.

Relativistic dynamics[a]

Lorentz factor	$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$	(3.69)	γ Lorentz factor $\boldsymbol{v}$ particle velocity c speed of light
Momentum	$\boldsymbol{p} = \gamma m_0 \boldsymbol{v}$	(3.70)	$\boldsymbol{p}$ relativistic momentum m_0 particle (rest) mass
Force	$\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}$	(3.71)	$\boldsymbol{F}$ force on particle t time
Rest energy	$E_{\mathrm{r}} = m_0 c^2$	(3.72)	E_{r} particle rest energy
Kinetic energy	$T = m_0 c^2(\gamma - 1)$	(3.73)	T relativistic kinetic energy
Total energy	$E = \gamma m_0 c^2$ $= (p^2c^2 + m_0^2 c^4)^{1/2}$	(3.74) (3.75)	E total energy $(= E_{\mathrm{r}} + T)$

[a] It is now common to regard mass as a Lorentz invariant property and to drop the term "rest mass." The symbol m_0 is used here to avoid confusion with the idea of "relativistic mass" $(= \gamma m_0)$ used by some authors.

Constant acceleration

$v = u + at$	(3.76)	u initial velocity
$v^2 = u^2 + 2as$	(3.77)	v final velocity
$s = ut + \frac{1}{2}at^2$	(3.78)	t time s distance travelled
$s = \frac{u+v}{2}t$	(3.79)	a acceleration

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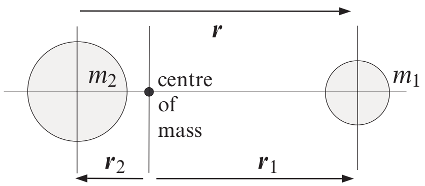
Reduced mass (of two interacting bodies)

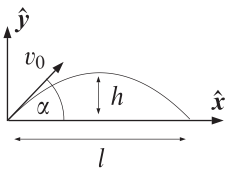


Reduced mass	$\mu = \frac{m_1 m_2}{m_1 + m_2}$	(3.80)	μ m_i	reduced mass interacting masses
Distances from centre of mass	$\boldsymbol{r}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{r}$	(3.81)	$\boldsymbol{r}_i$	position vectors from centre of mass
	$\boldsymbol{r}_2 = \frac{-m_1}{m_1 + m_2}\boldsymbol{r}$	(3.82)	$\boldsymbol{r}$ $\lvert\boldsymbol{r}\rvert$	$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ distance between masses
Moment of inertia	$I = \mu \lvert\boldsymbol{r}\rvert^2$	(3.83)	I	moment of inertia
Total angular momentum	$\boldsymbol{J} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$	(3.84)	$\boldsymbol{J}$	angular momentum
Lagrangian	$L = \frac{1}{2}\mu \lvert\dot{\boldsymbol{r}}\rvert^2 - U(\lvert\boldsymbol{r}\rvert)$	(3.85)	L U	Lagrangian potential energy of interaction

Ballistics[a]

Velocity	$\boldsymbol{v} = v_0 \cos\alpha\, \hat{\boldsymbol{x}} + (v_0 \sin\alpha - gt)\hat{\boldsymbol{y}}$	(3.86)	v_0 $\boldsymbol{v}$ α g	initial velocity velocity at t elevation angle gravitational acceleration
	$v^2 = v_0^2 - 2gy$	(3.87)		
Trajectory	$y = x\tan\alpha - \frac{gx^2}{2v_0^2\cos^2\alpha}$	(3.88)	$\hat{}$ t	unit vector time
Maximum height	$h = \frac{v_0^2}{2g}\sin^2\alpha$	(3.89)	h	maximum height
Horizontal range	$l = \frac{v_0^2}{g}\sin 2\alpha$	(3.90)	l	range

[a] Ignoring the curvature and rotation of the Earth and frictional losses. g is assumed constant.



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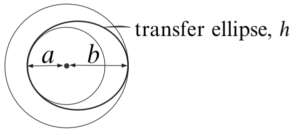
Rocketry

Escape velocity[a]	$v_{\mathrm{esc}} = \left(\frac{2GM}{r}\right)^{1/2}$	(3.91)
Specific impulse	$I_{\mathrm{sp}} = \frac{u}{g}$	(3.92)
Exhaust velocity (into a vacuum)	$u = \left[\frac{2\gamma R T_{\mathrm{c}}}{(\gamma-1)\mu}\right]^{1/2}$	(3.93)
Rocket equation ($g=0$)	$\Delta v = u \ln\left(\frac{M_{\mathrm{i}}}{M_{\mathrm{f}}}\right) \equiv u \ln \mathcal{M}$	(3.94)
Multistage rocket	$\Delta v = \sum_{i=1}^{N} u_i \ln \mathcal{M}_i$	(3.95)
In a constant gravitational field	$\Delta v = u \ln \mathcal{M} - g t \cos\theta$	(3.96)
Hohmann cotangential transfer[b]	$\Delta v_{ah} = \left(\frac{GM}{r_a}\right)^{1/2} \left[\left(\frac{2r_b}{r_a+r_b}\right)^{1/2} - 1\right]$	(3.97)
	$\Delta v_{hb} = \left(\frac{GM}{r_b}\right)^{1/2} \left[1 - \left(\frac{2r_a}{r_a+r_b}\right)^{1/2}\right]$	(3.98)

v_{esc}	escape velocity
G	constant of gravitation
M	mass of central body
r	central body radius
I_{sp}	specific impulse
u	effective exhaust velocity
g	acceleration due to gravity
R	molar gas constant
γ	ratio of heat capacities
T_{c}	combustion temperature
μ	effective molecular mass of exhaust gas
Δv	rocket velocity increment
M_{i}	pre-burn rocket mass
M_{f}	post-burn rocket mass
$\mathcal{M}$	mass ratio
N	number of stages
$\mathcal{M}_i$	mass ratio for ith burn
u_i	exhaust velocity of ith burn
t	burn time
θ	rocket zenith angle
Δv_{ah}	velocity increment, a to h
Δv_{hb}	velocity increment, h to b
r_a	radius of inner orbit
r_b	radius of outer orbit



[a] From the surface of a spherically symmetric, nonrotating body, mass M.
[b] Transfer between coplanar, circular orbits a and b, via ellipse h with a minimal expenditure of energy.

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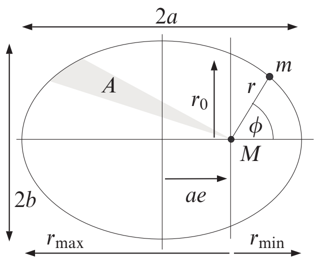
Gravitationally bound orbital motion[a]

Potential energy of interaction	$U(r) = -\frac{GMm}{r} \equiv -\frac{\alpha}{r}$	(3.99)	$U(r)$ G M m α	potential energy constant of gravitation central mass orbiting mass ($\ll M$) GMm (for gravitation)
Total energy	$E = -\frac{\alpha}{r} + \frac{J^2}{2mr^2} = -\frac{\alpha}{2a}$	(3.100)	E J	total energy (constant) total angular momentum (constant)
Virial theorem ($1/r$ potential)	$E = \langle U \rangle/2 = -\langle T \rangle$ $\langle U \rangle = -2\langle T \rangle$	(3.101) (3.102)	T $\langle \cdot \rangle$	kinetic energy mean value
Orbital equation (Kepler's 1st law)	$\frac{r_0}{r} = 1 + e\cos\phi$, or $r = \frac{a(1-e^2)}{1+e\cos\phi}$	(3.103) (3.104)	r_0 r e ϕ	semi-latus-rectum distance of m from M eccentricity phase (true anomaly)
Rate of sweeping area (Kepler's 2nd law)	$\frac{dA}{dt} = \frac{J}{2m} = \text{constant}$	(3.105)	A	area swept out by radius vector (total area $= \pi ab$)
Semi-major axis	$a = \frac{r_0}{1-e^2} = \frac{\alpha}{2\|E\|}$	(3.106)	a b	semi-major axis semi-minor axis
Semi-minor axis	$b = \frac{r_0}{(1-e^2)^{1/2}} = \frac{J}{(2m\|E\|)^{1/2}}$	(3.107)		
Eccentricity[b]	$e = \left(1 + \frac{2EJ^2}{m\alpha^2}\right)^{1/2} = \left(1 - \frac{b^2}{a^2}\right)^{1/2}$	(3.108)		
Semi-latus-rectum	$r_0 = \frac{J^2}{m\alpha} = \frac{b^2}{a} = a(1-e^2)$	(3.109)		
Pericentre	$r_{min} = \frac{r_0}{1+e} = a(1-e)$	(3.110)	r_{min}	pericentre distance
Apocentre	$r_{max} = \frac{r_0}{1-e} = a(1+e)$	(3.111)	r_{max}	apocentre distance
Speed	$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$	(3.112)	v	orbital speed
Period (Kepler's 3rd law)	$P = \pi\alpha\left(\frac{m}{2\|E\|^3}\right)^{1/2} = 2\pi a^{3/2}\left(\frac{m}{\alpha}\right)^{1/2}$	(3.113)	P	orbital period

[a]For an inverse-square law of attraction between two isolated bodies in the nonrelativistic limit. If m is not $\ll M$, then the equations are valid with the substitutions $m \to \mu = Mm/(M+m)$ and $M \to (M+m)$ and with r taken as the body separation. The distance of mass m from the centre of mass is then $r\mu/m$ (see earlier table on *Reduced mass*). Other orbital dimensions scale similarly, and the two orbits have the same eccentricity.

[b]Note that if the total energy, E, is <0 then $e<1$ and the orbit is an ellipse (a circle if $e=0$). If $E=0$, then $e=1$ and the orbit is a parabola. If $E>0$ then $e>1$ and the orbit becomes a hyperbola (see *Rutherford scattering* on next page).

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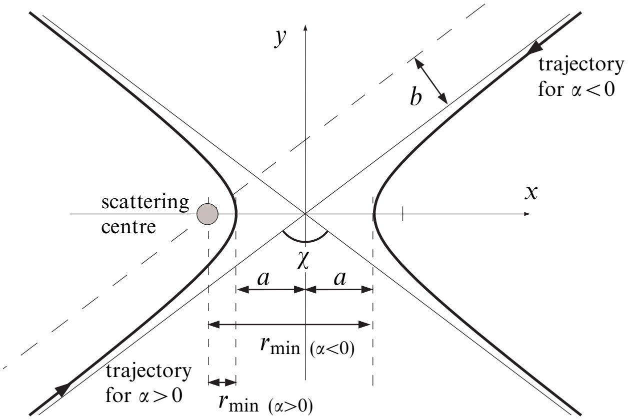
Rutherford scattering[a]



Scattering potential energy	$U(r)=-\frac{\alpha}{r}$	(3.114)	$U(r)$ potential energy; r particle separation; α constant
	$\alpha \begin{cases} <0 & \text{repulsive} \\ >0 & \text{attractive} \end{cases}$	(3.115)	
Scattering angle	$\tan\frac{\chi}{2}=\frac{\lvert\alpha\rvert}{2Eb}$	(3.116)	χ scattering angle; E total energy (>0); b impact parameter
Closest approach	$r_{\min}=\frac{\lvert\alpha\rvert}{2E}\left(\csc\frac{\chi}{2}-\frac{\alpha}{\lvert\alpha\rvert}\right)$	(3.117)	$r_{\min}$ closest approach; a hyperbola semi-axis; e eccentricity
	$=a(e\pm 1)$	(3.118)	
Semi-axis	$a=\frac{\lvert\alpha\rvert}{2E}$	(3.119)	
Eccentricity	$e=\left(\frac{4E^2b^2}{\alpha^2}+1\right)^{1/2}=\csc\frac{\chi}{2}$	(3.120)	
Motion trajectory[b]	$\frac{4E^2}{\alpha^2}x^2-\frac{y^2}{b^2}=1$	(3.121)	x,y position with respect to hyperbola centre
Scattering centre[c]	$x=\pm\left(\frac{\alpha^2}{4E^2}+b^2\right)^{1/2}$	(3.122)	
Rutherford scattering formula[d]	$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}=\frac{1}{n}\frac{\mathrm{d}N}{\mathrm{d}\Omega}$	(3.123)	$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ differential scattering cross section; n beam flux density; $\mathrm{d}N$ number of particles scattered into $\mathrm{d}\Omega$; Ω solid angle
	$=\left(\frac{\alpha}{4E}\right)^2\csc^4\frac{\chi}{2}$	(3.124)	

[a]Nonrelativistic treatment for an inverse-square force law and a fixed scattering centre. Similar scattering results from either an attractive or repulsive force. See also *Conic sections* on page 38.
[b]The correct branch can be chosen by inspection.
[c]Also the focal points of the hyperbola.
[d]n is the number of particles per second passing through unit area perpendicular to the beam.

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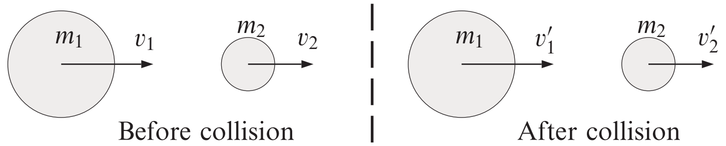
Inelastic collisions[a]

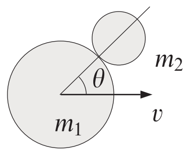


Coefficient of restitution	$v_2' - v_1' = \epsilon(v_1 - v_2)$ (3.125) $\epsilon = 1$ if perfectly elastic (3.126) $\epsilon = 0$ if perfectly inelastic (3.127)	ϵ v_i v_i'	coefficient of restitution pre-collision velocities post-collision velocities
Loss of kinetic energy[b]	$\frac{T - T'}{T} = 1 - \epsilon^2$ (3.128)	T, T'	total KE in zero momentum frame before and after collision
Final velocities	$v_1' = \frac{m_1 - \epsilon m_2}{m_1 + m_2} v_1 + \frac{(1+\epsilon) m_2}{m_1 + m_2} v_2$ (3.129) $v_2' = \frac{m_2 - \epsilon m_1}{m_1 + m_2} v_2 + \frac{(1+\epsilon) m_1}{m_1 + m_2} v_1$ (3.130)	m_i	particle masses

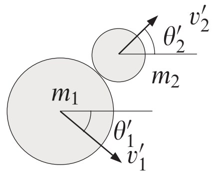
[a]Along the line of centres, $v_1, v_2 \ll c$.
[b]In zero momentum frame.

Oblique elastic collisions[a]

Before collision



After collision



Directions of motion	$\tan\theta_1' = \frac{m_2 \sin 2\theta}{m_1 - m_2 \cos 2\theta}$ (3.131) $\theta_2' = \theta$ (3.132)	θ θ_i' m_i	angle between centre line and incident velocity final trajectories sphere masses
Relative separation angle	$\theta_1' + \theta_2' \begin{cases} > \pi/2 & \text{if } m_1 < m_2 \\ = \pi/2 & \text{if } m_1 = m_2 \\ < \pi/2 & \text{if } m_1 > m_2 \end{cases}$ (3.133)		
Final velocities	$v_1' = \frac{(m_1^2 + m_2^2 - 2 m_1 m_2 \cos 2\theta)^{1/2}}{m_1 + m_2} v$ (3.134) $v_2' = \frac{2 m_1 v}{m_1 + m_2} \cos\theta$ (3.135)	v v_i'	incident velocity of m_1 final velocities

[a]Collision between two perfectly elastic spheres: m_2 initially at rest, velocities $\ll c$.

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